

# MCKAY CORRESPONDENCE FOR SEMISIMPLE HOPF ACTIONS ON REGULAR GRADED ALGEBRAS, II

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**ABSTRACT.** We continue our study of the McKay Correspondence for grading preserving actions of semisimple Hopf algebras  $H$  on (noncommutative) Artin-Schelter regular algebras  $A$ . Here, we establish correspondences between module categories over  $A^H$ , over  $A\#H$ , and over  $\text{End}_{A^H}A$ . We also study homological properties of (endomorphism rings of) maximal Cohen-Macaulay modules over  $A^H$ .

## 0. INTRODUCTION

Let  $\mathbb{k}$  be an algebraically closed field of characteristic zero. All algebraic structures in this article are over  $\mathbb{k}$ , and we take the unadorned  $\otimes$  to mean  $\otimes_{\mathbb{k}}$ . Throughout the introduction, let  $H$  be a semisimple Hopf algebra, and let  $A$  be an *Artin-Schelter* (AS) regular algebra  $A$  (Definition 1.1) that admits an action of  $H$ . By definition  $A$  is a connected graded algebra, and we assume the following throughout this paper:

**Hypothesis 0.1.** *Hopf algebra actions on graded algebras are grading preserving.*

The goal of this article is to continue Part I [7] of our work on the McKay Correspondence, extending results known for  $A = \mathbb{k}[x_1, \dots, x_n]$  and  $H = kG$  (for  $G$  a finite subgroup of  $\text{GL}_n(\mathbb{k})$ ) to the context of such Hopf actions on AS regular algebras.

Using appropriate definitions and hypotheses, which will follow, we achieve the results below. Here, let CM (respectively, MCM) stand for Cohen-Macaulay (respectively, maximal Cohen-Macaulay).

- (A) We establish bijections of indecomposable objects of various left module categories of  $H$ , of the smash product algebra  $A\#H$ , of the fixed subring  $A^H$ , and of the endomorphism ring  $\text{End}_{A^H}A$ .
- (B) We show that the McKay quiver and the Gabriel quiver associated to the  $H$ -action on  $A$  are isomorphic as directed graphs.
- (C) We produce a version of a theorem of Herzog [9] on indecomposable MCM modules over  $A^H$ , thus adding another correspondence to (A).

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(D, E) We also present a version of theorems of Leuschke [13] on the global dimension and on the representation dimension of endomorphism rings of MCM modules over  $A^H$ .

To begin, let us recall a conjecture that we posed in [7]: an analogue of Auslander's Theorem in the setting of semisimple Hopf actions on AS regular algebras. Auslander showed that when  $G$  contains no reflections (e.g. when  $G$  is a subgroup of  $\mathrm{SL}_n(\mathbb{k})$ ) and  $A = \mathbb{k}[x_1, \dots, x_n]$ , then there is a natural graded algebra isomorphism  $A \# G \cong \mathrm{End}_{A^G} A$ .

**Conjecture 0.2.** [7, Conjecture 0.1] *Let  $A$  be a noetherian AS regular algebra that admits an inner faithful action of a semisimple Hopf algebra  $H$ , with trivial homological determinant. Then there is a natural graded algebra isomorphism  $A \# H \cong \mathrm{End}_{A^H} A$ .*

The homological determinant is defined in Definition 1.5; in the classical setting (with  $H$  a group algebra and  $A$  a commutative polynomial ring) the homological determinant is the usual determinant of the linear map  $g \in G$ . This conjecture was also posed in [5, Conjecture 0.4]. In [7], we provided a partial resolution of this conjecture as follows.

**Theorem 0.3.** [7, Theorem 0.2] *If  $A$  is a noetherian AS regular algebra of dimension 2 generated in degree one, then Conjecture 0.2 is true.*

There are other cases where Conjecture 0.2 has been shown to hold, see [4, 5]. Now the precise statement of result (A), which depends on Conjecture 0.2, is given below. We call an  $A$ -module  $M$  an *initial* module if it is a graded module, generated in degree 0, with  $M_{<0} = 0$ .

**Theorem A** (Proposition 2.3, Corollary 2.6, and Lemma 3.17). *Let  $A$  be a noetherian AS regular algebra that admits an inner faithful action of a semisimple Hopf algebra  $H$ , with trivial homological determinant. If Conjecture 0.2 holds, then there are bijective correspondences between the isomorphism classes of:*

- (a) indecomposable direct summands of  $A$  as left  $A^H$ -modules;
- (b) indecomposable finitely generated, projective, initial left  $\mathrm{End}_{A^H} A$ -modules;
- (c) indecomposable finitely generated, projective, initial left  $A \# H$ -modules;
- and
- (d) simple left  $H$ -modules.

In the classical setting, the vertices of the quivers that arise naturally in the McKay Correspondence are the objects in the theorem above. In this direction, here, we provide an isomorphism of the *Gabriel quiver* (Definition 2.8) and the *McKay quiver* (Definition 2.9) of the  $H$ -action on  $A$ .

**Theorem B** (Theorem 2.10). *The Gabriel quiver  $\mathcal{G}(H, A)$  is isomorphic to the McKay quiver  $(W)\mathcal{M}$ .*

The last half of this work (Sections 3-5) pertains to CM and MCM modules (Definition 3.5(3,4)) over the fixed subring  $A^H$ . Result (C) is provided as follows.

**Theorem C** (Corollary 4.5). Let  $A$  be a noetherian AS regular algebra of dimension 2 that admits an action of a semisimple Hopf algebra  $H$ . Then there is a bijective correspondence between modules of Theorem A(a) and

- (e) indecomposable MCM left  $A^H$ -modules, up to a degree shift.

Finally, we draw our attention to endomorphism rings of MCM modules over algebras of *finite CM type* (Definition 5.2), particularly to achieve result **(D)**.

**Theorem D** (Theorem 5.4). Let  $B$  be a noetherian connected graded algebra with a balanced dualizing complex. Suppose that  $B$  is CM and of finite CM type. Let  $M$  be a finite direct sum of MCM right  $B$ -modules that contain at least one copy of each MCM module, up to a degree shift. Let  $E := \underline{\text{End}}_B(M)$ , the ring of graded endomorphisms of  $M$ .

- (1) If  $E$  is right noetherian, then  $E$  has right global dimension at most  $\max\{2, d\}$  where  $d = \text{cd}(B)$  (Definition 3.5).
- (2) If  $d \geq 2$ , then the right global dimension of  $E$  is equal to  $d$ .

Note that  $E$  in Theorem D (and Theorem E below) may not be  $\mathbb{N}$ -graded, but is always  $\mathbb{Z}$ -graded, bounded below, and locally finite. We introduce the notion of a *commonly graded* module (Definition 1.2(4)) to handle this situation. When  $\text{cd}(B) = 2$ , we have a stronger result.

**Theorem E** (Theorem 5.7). Let  $B$  be a noetherian connected graded algebra that is CM with  $\text{cd}(B) = 2$ , of finite CM type, and module-finite over a central affine graded subalgebra  $C$ . Let  $M$  be a finite direct sum of MCM right  $B$ -modules that contains at least one copy of each MCM module, up to a degree shift. Then  $E := \underline{\text{End}}_B(M)$  is a noetherian PI generalized AS regular algebra (Definition 3.9) of dimension 2.

The paper is organized as follows. Section 1 reviews required definitions and results pertaining to AS regular algebras, Hopf algebra actions, and module categories. Section 2 contains some of the results on the module category correspondences and the Gabriel and McKay quivers. Section 3 provides results on local cohomology and the Cohen-Macaulay conditions, including a number of preliminary results extended to the commonly graded setting. Section 4 contains the generalization of Herzog's Theorem, and Section 5 contains the generalizations of Leuschke's theorems.

## 1. PRELIMINARIES

In this section we briefly review background material on Artin-Schelter regular algebras, Hopf algebra actions on graded algebras, and some abelian categories of (certain) modules over a given algebra.

Recall that  $\underline{\otimes}$  (respectively,  $\underline{\text{Tor}}$ ,  $\underline{\text{Hom}}$ ,  $\underline{\text{End}}$ ,  $\underline{\text{Ext}}$ ) denotes the grading preserving operation  $\otimes$  (respectively,  $\text{Tor}$ ,  $\text{Hom}$ ,  $\text{End}$ ,  $\text{Ext}$ ).

**1.1. Artin-Schelter regular algebras.** An algebra  $A$  is said to be *connected graded* if  $A = \mathbb{k} \oplus A_1 \oplus A_2 \oplus \cdots$  with  $A_i \cdot A_j \subseteq A_{i+j}$  for all  $i, j \in \mathbb{N}$ . We consider a class of noncommutative graded algebras that serve as noncommutative analogues of commutative polynomial rings. These algebras are defined as follows.

**Definition 1.1.** A connected graded algebra  $A$  is called *Artin-Schelter (AS) Gorenstein* if the following conditions hold:

- (a)  $A$  has finite injective dimension  $d < \infty$  on both sides,
- (b)  $\text{Ext}_A^i(\mathbb{k}, A) = \text{Ext}_{A^{\text{op}}}^i(\mathbb{k}, A) = 0$  for all  $i \neq d$  where  $\mathbb{k} = A/A_{\geq 1}$ , and
- (c)  $\text{Ext}_A^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$  and  $\text{Ext}_{A^{\text{op}}}^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$  for some integer  $\ell$ .

The integer  $\ell$  is called the *AS index* of  $A$ . If moreover,

- (d)  $A$  has finite global dimension  $d$ ,
- (e)  $A$  has finite Gelfand-Kirillov dimension,

then  $A$  is called *Artin-Schelter (AS) regular* of dimension  $d$ .

AS regular algebras of global dimension 2 are given in [7, Example 1.2]. Even if  $A$  is connected graded, graded  $A$ -modules need not be  $\mathbb{N}$ -graded. We introduce the following.

**Definition 1.2.** Let  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  be a  $\mathbb{Z}$ -graded module (or  $\mathbb{k}$ -vector space or an algebra).

- (1) The *lower bound* of  $M$  is defined to be

$$b_l(M) = \min\{i \mid M_i \neq 0\};$$

$M$  is called *bounded below* if  $b_l(M) > -\infty$ .

- (2) The *upper bound* of  $M$  is defined to be

$$b_u(M) = \max\{i \mid M_i \neq 0\};$$

$M$  is called *bounded above* if  $b_u(M) < \infty$ .

- (3)  $M$  is called *locally finite* if  $\dim_{\mathbb{k}} M_i < \infty$  for every  $i$ .
- (4)  $M$  is called *commonly graded* if it is bounded below and locally finite.

**1.2. Hopf algebra actions.** Throughout this work  $H$  stands for a Hopf algebra over  $\mathbb{k}$  with structural notation  $(H, m, u, \Delta, \epsilon, S)$ . We use Sweedler notation; namely,  $\Delta(h) = \sum h_1 \otimes h_2$ . The action of  $H$  on an algebra  $A$  is given as follows.

**Definition 1.3.** Let  $A$  be an algebra and  $H$  a Hopf algebra.

- (1) We say that  $A$  is a (*left*)  *$H$ -module algebra*, or that  $H$  *acts on*  $A$ , if  $A$  is an algebra in the category of left  $H$ -modules. Equivalently,  $A$  is a left  $H$ -module such that  $h(ab) = \sum h_1(a)h_2(b)$  and  $h(1_A) = \epsilon(h)1_A$  for all  $h \in H$ , and all  $a, b \in A$ .
- (2) We say that  $A$  is a *graded  $H$ -module algebra* if  $A$  is an algebra in the category of  $\mathbb{Z}$ -graded (left)  $H$ -modules (with elements in  $H$  having degree zero), or equivalently, each homogeneous component of  $A$  is a left  $H$ -submodule.
- (3) Given an  $H$ -module algebra, we form the *smash product algebra*  $A \# H$ , which is equal to  $A \otimes H$  as a  $\mathbb{k}$ -vector space, and the multiplication in  $A \# H$  is given by  $(a \# h)(a' \# h') = \sum ah_1(a') \# h_2h'$  for all  $h, h' \in H$  and  $a, a' \in A$ .

We identify  $H$  with a subalgebra of  $A\#H$  via the map  $i_H : h \rightarrow 1\#h$  for all  $h \in H$ , and we identify  $A$  with a subalgebra of  $A\#H$  via the map  $i_A : a \rightarrow a\#1$  for all  $a \in A$ . If  $A$  is a graded  $H$ -module algebra, then  $A\#H$  is graded with  $\deg h = 0$  for all  $0 \neq h \in H$  and  $A$  is a graded subalgebra of  $A\#H$ . If  $A$  is commonly graded and  $H$  is finite dimensional, then  $A\#H$  is commonly graded and  $(A\#H)_i = A_i\#H$  for all  $i$ .

Sometimes it is useful to restrict ourselves to Hopf  $(H)$ -actions that do not factor through the action of a proper Hopf quotient of  $H$ .

**Definition 1.4.** Let  $M$  be a left  $H$ -module. We say that  $M$  is an *inner faithful*  $H$ -module, or  $H$  *acts inner faithfully* on  $M$ , if  $IM \neq 0$  for every nonzero Hopf ideal  $I$  of  $H$ . The same terminology applies to  $H$ -module algebras  $A$ .

Moreover, the homological determinant of a Hopf algebra action on an Artin-Schelter algebra is given below. Recall that a connected graded algebra  $A$  is AS regular if and only if the Ext-algebra  $E(A) := \bigoplus_{i \geq 0} \text{Ext}_A^i(A\mathbb{k}, A\mathbb{k})$  of  $A$  is Frobenius [16, Corollary D].

**Definition 1.5.** Retain the notation above. Let  $A$  be an AS regular algebra with Frobenius Ext-algebra  $E$ . Suppose  $\epsilon$  is a nonzero element in  $\text{Ext}_A^d(A\mathbb{k}, A\mathbb{k})$ , where  $d = \text{gldim}(A)$ . Let  $H$  be a Hopf algebra acting on  $A$  from the left, and hence  $H$  acts on  $E$  from the left.

- (1) The *homological determinant* of the  $H$ -module algebra  $A$  is defined to be  $\eta \circ S$ , where  $\eta : H \rightarrow k$  is determined by  $h \cdot \epsilon = \eta(h)\epsilon$ .
- (2) The homological determinant is *trivial* if  $\text{hdet}_H A = \epsilon$ .

**1.3. Module categories.** For an algebra  $A$ , consider the following categories of modules over  $A$ .

- $A\text{-Mod}$  (respectively,  $\text{Mod-}A$ ): the category of all left (respectively, right)  $A$ -modules.
- $A\text{-Prm}$  (respectively,  $\text{Prm-}A$ ): the full subcategory of  $A\text{-Mod}$  (respectively,  $\text{Mod-}A$ ) consisting of projective left (respectively, right)  $A$ -modules.
- $A\text{-Grm}$  (respectively,  $\text{Grm-}A$ ), when  $A$  is graded: the category of  $\mathbb{Z}$ -graded left (respectively, right)  $A$ -modules.
- $A\text{-Grprm}$  (respectively,  $\text{Grprm-}A$ ), when  $A$  is graded: the full subcategory of  $A\text{-Grm}$  (respectively,  $\text{Grm-}A$ ) consisting of projective left graded (respectively, right)  $A$ -modules.
- $A\text{-xyz}$ , when  $A\text{-Xyz}$  is a category of left  $A$ -modules with property  $\text{Xyz}$ : the full subcategory of  $A\text{-Xyz}$  consisting of *finite* (that is, finitely generated) modules in  $A\text{-Xyz}$ .
- $A\text{-Xyz}_0$ , when  $A$  is graded and  $A\text{-Xyz}$  is a category of left  $A$ -modules with property  $\text{Xyz}$ : the full subcategory of  $A\text{-Xyz}$  consisting of *initial* modules, that is,  $M \in A\text{-Xyz}$  generated in degree 0 with  $M_{<0} = 0$ .
- $\text{add}_{A\text{-grm}} M$ , when  $A$  is graded and  $M$  is a finite graded left  $A$ -module: the full subcategory of  $A\text{-grm}$  containing all direct summands of finite direct sums of degree shifts of  $M$ .

- $\text{xyz-}A$ ,  $\text{Xyz}_0-A$ , and  $\text{add}_{\text{grm-}A} M$ , are defined likewise for right  $A$ -modules.

Unless otherwise stated, we work with left modules. However, in the notation  $\text{End}_{A^H}(A)$  (or  $\text{End}_B M$ ),  $A$  (or  $M$ ) is considered as a right  $A^H$ -module (or a right  $B$ -module), as it was in [7]. Sometimes it is easy to switch from the left to the right as the next well-known lemma shows. A contravariant equivalence between two categories is called a *duality*.

**Lemma 1.6.** *Let  $A$  be an algebra. Then there is a duality of categories*

$$\text{Hom}_A(-, A_A) : \text{prm-}A \rightarrow A\text{-prm}.$$

*As a consequence, there is a bijection between the isomorphism classes of indecomposable finite projective left  $A$ -modules and that of indecomposable finite projective right  $A$ -modules.*  $\square$

## 2. CORRESPONDENCES BETWEEN MODULE CATEGORIES

Let  $H$  be a semisimple Hopf algebra and  $A$  an  $H$ -module algebra. In this section we establish several equivalences between categories of modules over  $H$ , over the smash product algebra  $A\#H$ , and over the endomorphism ring  $\text{End}_{A^H} A$ . Some of the proofs are generalizations of results in the commutative setting, and a good review of the commutative case can be found in [14]. We also define the Gabriel quiver in our setting (Definition 2.8) and establish that it is isomorphic to the McKay quiver (Definition 2.9) studied in [7].

**2.1. Projective modules over the smash product algebra.** We begin with a useful lemma that was proved in [8].

**Lemma 2.1.** *Let  $H$  be a semisimple Hopf algebra acting on an algebra  $A$ . Let  $M$  and  $N$  be left  $A\#H$ -modules. Then the following statements hold.*

- (1) *For every  $i \geq 0$ ,  $\text{Ext}_{A\#H}^i(M, N) = \text{Ext}_A^i(M, N)^H$ .*
- (2)  *$M$  is projective over  $A\#H$  if and only if it is projective over  $A$ .*

*Proof.* Part (1) is [8, Corollary 2.17].

(2) If  $M$  is projective over  $A\#H$ , there is an  $A\#H$ -module  $Q$  such that  $M \oplus Q$  is free over  $A\#H$ . Since  $A\#H$  is free over  $A$ , then  $M \oplus Q$  is free over  $A$ . Hence,  $M$  is projective over  $A$ . The converse follows directly from part (1).  $\square$

Next we study the category of graded modules over  $A\#H$  when  $A$  is a *connected graded* algebra and  $H$  is a semisimple Hopf algebra acting on  $A$ . Note that  $A\#H$  is a graded algebra extension of  $A$  with  $\deg h = 0$  for all  $h \in H$ .

**Lemma 2.2.** *Retaining the notation above, let  $M$  be a commonly graded left  $A\#H$ -module. Then the following statements hold.*

- (1) *The projective cover of  $M$  is  $A \otimes (M/\mathfrak{m}M)$ , where  $\mathfrak{m}$  is the maximal graded ideal of  $A$ .*
- (2) *If  $M$  is generated in degree  $i$ , then the projective cover of  $M$  is  $(A \otimes M_i)[-i]$ .*

*Proof.* (1) Since  $H$  is semisimple,  $M/\mathfrak{m}M$  is isomorphic to a graded  $H$ -submodule of  $M$ , say  $M_0$ , that generates  $M$  as a left  $A$ -module. Then  $A \otimes M_0$  is a graded left  $A\#H$ -module determined by

$$(E2.2.1) \quad (x\#h)(a \otimes m) = \sum xh_1(a) \otimes h_2(m)$$

for all  $x, a \in A$ ,  $h \in H$  and  $m \in M_0$ . Note that  $A \otimes M_0$  is a free left  $A$ -module. By Lemma 2.1(2),  $A \otimes M_0$  is a graded projective  $A\#H$ -module. Define a map

$$\phi : A \otimes M_0 \rightarrow M \quad \text{by} \quad \phi(a \otimes m) = am$$

for all  $a \in A$  and  $m \in M_0$ . It is easy to check that this map is a surjective  $A\#H$ -module morphism, and  $A \otimes M_0$  is a projective cover of  $M$  as a left  $A$ -module because  $(A/\mathfrak{m}) \otimes_A \phi$  is an isomorphism by the choice of  $M_0$ . Therefore  $A \otimes (M/\mathfrak{m}M)$  is a projective cover of  $M$  as a left  $A\#H$ -module.

Clearly (2) follows from (1).  $\square$

This lemma prompts the following proposition.

**Proposition 2.3.** *Let  $H$  be a semisimple Hopf algebra acting on a connected graded algebra  $A$ . Then we have the following equivalences of categories:*

$$(E2.3.1) \quad A\#H\text{-Grpm} \simeq H\text{-Grm},$$

$$(E2.3.2) \quad A\#H\text{-Grpm}_0 \simeq H\text{-Mod}.$$

*Proof.* We first prove the equivalence (E2.3.1). Let  $\mathfrak{m}$  be the maximal graded ideal of  $A$ , and we define the functors

$$\begin{aligned} \Phi : A\#H\text{-Grpm} &\rightarrow H\text{-Grm}, & P &\mapsto P/\mathfrak{m}P = A/\mathfrak{m} \otimes_A P \\ \Psi : H\text{-Grm} &\rightarrow A\#H\text{-Grpm}, & M &\mapsto A \otimes M. \end{aligned}$$

Clearly  $\Phi$  maps into  $H\text{-Grm}$ . To show that  $\Psi$  maps into  $A\#H\text{-Grpm}$ , let  $M$  be a left graded  $H$ -module. Then  $A \otimes M$  is a graded left  $A\#H$ -module via (E2.2.1) (by replacing  $M_0$  with  $M$ ). By Lemma 2.1(2),  $A \otimes M_i$  is a graded projective  $A\#H$ -module for each  $i$ . So  $A \otimes M$  is a graded projective  $A\#H$ -module.

Since  $A/\mathfrak{m} \cong \mathbb{k}$ , we have

$$\Phi \circ \Psi(M) = \Phi(A \otimes M) = \mathbb{k} \otimes M \cong M,$$

which implies that  $\Phi \circ \Psi$  is naturally isomorphic to the identity functor of  $H\text{-Grm}$ . For any  $P$  in  $A\#H\text{-Grpm}$ ,  $P/\mathfrak{m}P$  generates  $P$  by the graded Nakayama Lemma. So we can define a surjective map

$$\gamma : A \otimes (P/\mathfrak{m}P) \rightarrow P, \quad a \otimes p \mapsto ap.$$

Since  $A \otimes (P/\mathfrak{m}P)$  is the projective cover of  $P$  by Lemma 2.2(1), and since  $P$  is projective, it follows that  $\gamma$  is an isomorphism. Therefore  $\Psi \circ \Phi$  is naturally isomorphic to the identity functor of  $A\#H\text{-Grpm}$ .

The equivalence (E2.3.2) holds by restricting the equivalence (E2.3.1) to the initial graded modules over  $A\#H$ .  $\square$

Let  $A$  be a connected graded algebra. Then we have a minimal free resolution of the left trivial  $A$ -module  $\mathbb{k}$ :

$$(E2.3.3) \quad \cdots \rightarrow A \otimes Q_n \xrightarrow{\phi_n} A \otimes Q_{n-1} \rightarrow \cdots \rightarrow A \otimes Q_1 \xrightarrow{\phi_1} A \rightarrow \mathbb{k} \rightarrow 0$$

where each  $Q_i$  is a graded  $\mathbb{k}$ -vector space and each map  $\phi_n$  has entries in  $\mathfrak{m}$ . If  $A$  is generated in degree one, then  $Q_1$  is in degree one.

**Lemma 2.4.** *Let  $A$  be a connected graded algebra admitting an action of a semisimple Hopf algebra  $H$ . Let  $M$  be a graded bounded below left  $A\#H$ -module. Then there is a minimal projective resolution of the  $A\#H$ -module  $M$  that is also a minimal free resolution of left  $A$ -module  $M$ . As a result, (E2.3.3) can be considered as a minimal projective resolution of  $\mathbb{k}$  as an  $A\#H$ -module.*

*Proof.* By Lemmas 2.1 and 2.2,  $P_0 := A \otimes (M/\mathfrak{m}M)$  is a projective cover of  $M$ , which is also a projective cover of  $M$  as a graded  $A$ -module. Let  $M_1 = \ker(P_0 \rightarrow M)$ . Then we have a short exact sequence of graded left  $A\#H$ -modules

$$0 \rightarrow M_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

By taking the projective cover of  $M_1$ , we obtain  $P_1$ , which is also the projective cover of  $M_1$  as a graded left  $A$ -module. The standard process of constructing a minimal projective resolution leads to the result.  $\square$

**2.2. Modules over endomorphism rings.** Let  $B$  be a graded algebra and let  $M$  be a finite graded right  $B$ -module. We present below a graded version of a result in [3] that will be used to relate projective modules over the endomorphism ring of  $M$  to  $\text{add}_{\text{grm-}B} M$ ; the proof is a straight-forward adaptation of that in [3].

**Proposition 2.5.** [3, Proposition II.2.1] *Let  $B$  be a commonly graded algebra and let  $M$  be a finite graded right  $B$ -module. Denote by  $E$  the ring  $\underline{\text{End}}_{\text{grm-}B} M$  of graded endomorphisms of  $M$ . Then the evaluation functor*

$$e_M := \underline{\text{Hom}}_{\text{grm-}B}(M, -) : \text{grm-}B \longrightarrow \text{grm-}E$$

*has the following properties.*

(1) *We have that*

$$\underline{\text{Hom}}_{\text{grm-}B}(Y, Z) \cong \underline{\text{Hom}}_{\text{grm-}E}(e_M(Y), e_M(Z))$$

*for  $Y \in \text{add}_{\text{grm-}B} M$  and  $Z \in \text{grm-}B$ .*

(2) *If  $X$  is in  $\text{add}_{\text{grm-}B} M$ , then  $e_M(X)$  is in  $\text{grprpm-}E$ .*

(3)  *$e_M|_{\text{add}_{\text{grm-}B} M}$  induces an equivalence of categories*

$$\text{add}_{\text{grm-}B} M \simeq \text{grprpm-}E. \quad \square$$

Now the results above yield, in our context of semisimple Hopf actions on AS regular algebras, part of the McKay Correspondence pertaining to module categories over  $H$ , over  $A\#H$ , and over  $\text{End}_{A\#H} A$ .

**Corollary 2.6.** *Let  $A$  be a noetherian AS regular algebra and  $H$  be a semisimple Hopf algebra that acts on  $A$  inner faithfully so that Conjecture 0.2 holds. Then there are natural bijections between isomorphism classes of*



- (a) *indecomposable objects in  $A\#H$ -grprm,*
- (b) *indecomposable objects in  $\underline{\text{End}}_{A^H}A$ -grprm,*
- (c) *indecomposable right  $A^H$ -modules in  $\text{add}_{\text{grm-}A^H}A$ .*

*Proof.* Conjecture 0.2 gives a graded algebra isomorphism  $A\#H \cong \underline{\text{End}}_{A^H}A$ , which induces a bijection between (a) and (b). Proposition 2.5(3) for  $M = A$  and  $B = A^H$  induces an equivalence

$$\text{add}_{\text{grm-}A^H}A \simeq \text{grprm-}\underline{\text{End}}_{A^H}A.$$

In particular, this equivalence gives a bijection between the indecomposable objects in the respective categories. Combining these facts with Lemma 1.6 gives a bijection between (b) and (c).  $\square$

**2.3. The Gabriel and McKay quivers.** Let us consider the following notation that will be used throughout this subsection.

*Notation.* Let  $H$  be a semisimple Hopf algebra that acts on a connected graded algebra  $A$  so that  $A$  is finitely generated in degree one by a left  $H$ -module  $W$ . Moreover, let  $\{V^{(0)}, V^{(1)}, \dots, V^{(s)}\}$  be a complete set of nonisomorphic simple left  $H$ -modules, where  $V^{(0)} = H/(\ker \epsilon)$ , and let  $P^{(j)}$  be the left  $A\#H$ -module  $A \otimes V^{(j)}$  as defined by (E2.2.1).

The result below holds by Proposition 2.3 and Lemma 2.4, using  $H$  and the maximal graded ideal of  $A$  in the roles of  $\mathbb{k}G$  and  $\mathfrak{m}$ , respectively, in [2, Section 1]. See, also, [15, Corollary 5.19].

**Lemma 2.7.** *Let  $j$  be an integer between 0 and  $s$ .*

- (1) *The complete set of nonisomorphic, indecomposable, initial, projective left  $A\#H$ -modules is  $\{P^{(0)}, P^{(1)}, \dots, P^{(s)}\}$ .*
- (2) *The module  $V^{(j)}$  is a simple left  $A\#H$ -module via the surjection  $A\#H \rightarrow H \rightarrow V^{(j)}$ , with projective cover  $P^{(j)}$ .*
- (3) *Suppose*

$$(E2.7.1) \quad \dots \rightarrow Q_n^{(j)} \rightarrow \dots \rightarrow Q_1^{(j)} \rightarrow Q_0^{(j)} \rightarrow V^{(j)} \rightarrow 0$$

*is a minimal projective resolution of the  $A\#H$ -module  $V^{(j)}$ , then the minimal projective resolution of  $V^{(j)}$  over  $A$  also has this form.*

*Proof.* (1,2) Clear.

- (3) This follows from Lemma 2.4.  $\square$

Now we define the quivers of interest.

**Definition 2.8.** The *Gabriel quiver*  $\mathcal{G}(H, A)$  of the  $H$ -action on  $A$  is the directed graph with vertices  $P^{(0)}, P^{(1)}, \dots, P^{(s)}$ , and with  $m_{ij}$  arrows from  $P^{(i)}$  to  $P^{(j)}$ , where  $m_{ij}$  is the multiplicity of  $P^{(i)}[-1]$  in  $Q_1^{(j)}$  as defined in (E2.7.1).

**Definition 2.9.** [7, Definition 2.3] Let  $W$  be a finite dimensional left  $H$ -module. The *McKay quiver*  $(W)\mathcal{M}$  has vertices  $V^{(0)}, V^{(1)}, \dots, V^{(s)}$ , and  $n_{ij}$  arrows from  $V^{(i)}$  to  $V^{(j)}$  where  $W \otimes V^{(j)} = \bigoplus_{i=0}^s (V^{(i)})^{\oplus n_{ij}}$ . The set of values  $n_{ij}$  are referred to as the *fusion rule coefficients*.

The following result, our Theorem B, was proved by Auslander [2, Section 1] when  $A$  is a commutative polynomial ring or a formal power series ring and  $H$  is a group algebra.

**Theorem 2.10.** *The Gabriel quiver  $\mathcal{G}(H, A)$  is isomorphic to the McKay quiver  $(W)\mathcal{M}$ .*

*Proof.* By Proposition 2.3, the number of vertices of  $\mathcal{G}(H, A)$  and of  $(W)\mathcal{M}$  are the same; namely,  $P^{(j)}$  corresponds to  $V^{(j)}$  for  $j = 0, \dots, s$ . So it suffices to show that the values  $m_{ij}$  in Definition 2.8 form the fusion rule coefficients of Definition 2.9.

Write a minimal projective resolution of the trivial  $A$ -module  $V^{(0)} = \mathbb{k}$  as

$$\cdots \rightarrow A \otimes W_n \rightarrow \cdots \rightarrow A \otimes W_2 \rightarrow A \otimes W[-1] \rightarrow A \rightarrow \mathbb{k} \rightarrow 0,$$

which is also a minimal projective resolution of  $\mathbb{k}$  as a graded left  $A\#H$ -module by Lemma 2.7(3). Tensoring with  $V^{(j)}$  from the right yields a minimal projective resolution of the left  $A\#H$ -module  $V^{(j)}$ ,

$$\cdots \rightarrow \cdots \rightarrow (A \otimes W_2) \otimes V^{(j)} \rightarrow (A \otimes W[-1]) \otimes V^{(j)} \rightarrow A \otimes V^{(j)} \rightarrow V^{(j)} \rightarrow 0.$$

Note that  $A \otimes V^{(j)} = P^{(j)}$  and  $(A \otimes W[-1]) \otimes V^{(j)} = Q_1^{(j)}$  as in Lemma 2.7. We obtain, by Definitions 2.8 and 2.9,

$$\begin{aligned} \bigoplus_{i=0}^s \left( P^{(i)}[-1] \right)^{\oplus m_{ij}} &\cong Q_1^{(j)} \\ &= (A \otimes W[-1]) \otimes V^{(j)} \cong A \otimes (W[-1] \otimes V^{(j)}) \\ &\cong A \otimes \left( \bigoplus_{i=0}^s (V^{(i)})^{\oplus n_{ij}} \right) [-1] \\ &\cong \bigoplus_{i=0}^s \left( P^{(i)}[-1] \right)^{\oplus n_{ij}}. \end{aligned}$$

Therefore,  $m_{ij} = n_{ij}$  for all  $i, j$ , as desired.  $\square$

### 3. LOCAL COHOMOLOGY AND THE COHEN-MACAULAY PROPERTY

In this section we recall some basic definitions related to local cohomology and Cohen-Macaulay modules needed for the results in Sections 4 and Section 5. We refer to [1, 10, 20, 23] for details and some undefined terminology. We will also extend a number of results from the connected graded to the commonly graded setting.

**Definition 3.1.** Let  $A$  be a commonly graded algebra (Definition 1.2(4)), which is  $\mathbb{Z}$ -graded, but may not be  $\mathbb{N}$ -graded.

- (1) [17, Definition 1.7.3] The *graded Jacobson radical* of  $A$  is

$$\mathfrak{m}_A := \bigcap \{ L \mid L \text{ maximal left graded ideals of } A \}.$$

- (2)  $A$  is called *graded semilocal* if  $A/\mathfrak{m}_A$  is graded semisimple ([17, Section 1.7]).  
 (3) For a graded left  $A$ -module  $M$ , the *graded  $\mathbb{k}$ -linear dual* of  $M$  is

$$M^* := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}(M_{-i}, \mathbb{k}).$$

It is easy to see that  $M^*$  is a graded right  $A$ -module.

We start with an easy lemma.

**Lemma 3.2.** *Let  $A$  be a commonly graded algebra.*

- (1) *If  $L$  is a left maximal graded ideal of  $A$ , then  $L \supseteq \bigoplus_{i > -b_l(A)} A_i$ . As a consequence,  $\mathfrak{m}_A \supseteq \bigoplus_{i > -b_l(A)} A_i$ .*
- (2) *If  $S$  is a factor graded simple left  $A$ -module of  $A$ , then  $b_l(S) \geq b_l(A)$  and  $b_u(S) \leq -b_l(A)$ . As a consequence,  $S$  is finite dimensional.*
- (3) *There are only finitely many isomorphism classes of simple graded left  $A$ -modules, up to degree shifts. As a consequence,  $A$  is graded semilocal.*
- (4) [17, Lemma 1.7.4(4)]

$$\mathfrak{m}_A = \bigcap \{l.\text{ann}(S) \mid S \text{ graded simple left } A\text{-modules}\}.$$

- (5) [17, Lemma 1.7.4(7)]

$$\begin{aligned} \mathfrak{m}_A &= \bigcap \{R \mid R \text{ maximal right graded ideals of } A\} \\ &= \bigcap \{l.\text{ann}(S) \mid S \text{ graded simple right } A\text{-modules}\}. \end{aligned}$$

- (6) *If  $M$  is a graded left  $A$ -module, then there is a natural isomorphism*

$$M^* \cong \underline{\text{Hom}}_A(M, A^*).$$

*As a consequence,  $A^*$  is a graded injective left (or right)  $A$ -module.*

*Proof.* (1) Let  $x \in \bigoplus_{i > -b_l(A)} A_i$ . Then  $b_l(xA) \geq \deg x + b_l(A) > 0$  and  $(xA + L)_0 = L_0$ . Then  $xA + L \neq A$ . Since  $L$  is maximal,  $xA + L = L$  or  $xA \subseteq L$ . Thus  $x \in L$  and consequently,  $\bigoplus_{i > -b_l(A)} A_i \subseteq L$ .

(2) This follows from part (1) and the fact that  $S = A/L$  for some maximal graded left ideal  $L$ .

(3) Let  $S$  be a graded simple left  $A$ -module. Then  $\mathfrak{m}_A S = 0$ . Thus  $S$  is a factor module of  $A/\mathfrak{m}_A$  up to a shift. Since  $A/\mathfrak{m}_A$  is finite dimensional by part (1), there are only finitely many isomorphism classes of graded simple left  $A$ -modules, up to a shift.

(4,5) These are properties of the graded Jacobson radical.

(6) For every graded left  $A$ -module  $M$ , one has

$$\begin{aligned} M^* &= \underline{\text{Hom}}_{\mathbb{k}}(M, \mathbb{k}) && \cong \underline{\text{Hom}}_{\mathbb{k}}(M \otimes_A A, \mathbb{k}) \\ &\cong \underline{\text{Hom}}_A(M, \underline{\text{Hom}}_{\mathbb{k}}(A, \mathbb{k})) && = \underline{\text{Hom}}_A(M, A^*); \end{aligned}$$

see [17, Section 1.2]. Since the functor  $(-)^*$  is exact,  $A^*$  is an injective graded left  $A$ -module.  $\square$

Part (2) of the following is a graded version of Nakayama lemma (see, also, [17, Lemma 1.7.5]).

**Proposition 3.3.** *Let  $A$  be a commonly graded algebra.*

- (1) *Suppose  $A$  is left noetherian. For each  $i > 0$ , there are  $j$  and  $k$  such that*

$$A_{\geq k} \subseteq (\mathfrak{m}_A)^j \subseteq A_{\geq i}.$$

*As a consequence, the two sequences  $\{\mathfrak{m}_A^n\}_n$  and  $\{A_{\geq n}\}_n$  are cofinal.*

- (2) *Let  $M$  be a bounded below left  $A$ -module. If  $\mathfrak{m}_A M = M$ , then  $M = 0$ .*

*Proof.* (1) Let  $I$  be the ideal  $A(A_{\geq 1-2b_l(A)})A$  which is a subspace of  $A_{\geq 1}$ . Let  $B = A/I$ . Then  $B$  is a finite dimensional graded algebra, and  $(\mathfrak{m}_B)^N = 0$  for some integer  $N > 0$ . This means that  $(\mathfrak{m}_A)^N \subseteq I \subseteq A_{\geq 1}$ . Therefore  $(\mathfrak{m}_A)^{iN} \subseteq A_{\geq i}$  for all  $i$ , and so we can take  $j = iN$ .

Now we assume that  $A$  is left noetherian. Then for each  $j$ ,  $A/(\mathfrak{m}_A^j)$  is finite dimensional. This means that  $A_{\geq k} \subseteq \mathfrak{m}_A^j$  for some  $k$ .

(2) By the proof of part (1), even without the left noetherian hypothesis, one has  $\mathfrak{m}_A^j \subseteq A_{\geq i}$ .

Suppose that  $M \neq 0$  and that  $\mathfrak{m}_A M = M$ . Then  $\mathfrak{m}_A^n M = M$  for all  $n \gg 0$ . By part (1), we can choose  $n$  such that  $\mathfrak{m}_A^n \subseteq A_{\geq 1}$ . Then

$$b_l(M) = b_l(\mathfrak{m}_A^n M) \geq b_l(\mathfrak{m}_A^n) + b_l(M) \geq 1 + b_l(M) > b_l(M),$$

yielding a contradiction. Therefore  $M = 0$ .  $\square$

Let  $A$  be a left noetherian commonly graded algebra and  $\mathfrak{m}_A$  (or  $\mathfrak{m}$  if no confusion occurs) be the graded Jacobson radical of  $A$ . For each graded left  $A$ -module  $M$ , the  $\mathfrak{m}_A$ -torsion submodule of  $M$  is defined to be

$$\Gamma_{\mathfrak{m}_A}(M) := \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n \rightarrow \infty} \underline{\text{Hom}}_A(A/(\mathfrak{m}_A)^n, M).$$

The functor  $\Gamma_{\mathfrak{m}_A}$  is a left exact functor from  $A\text{-Grm}$  to itself.

We now extend some definitions to the commonly graded case.

**Definition 3.4.** The  $i$ -th right derived functor  $R^i\Gamma_{\mathfrak{m}_A}(-)$  is called the  $i$ -th local cohomology, and  $R^i\Gamma_{\mathfrak{m}_A}(M)$  is called the  $i$ -th local cohomology module of  $M$ .

**Definition 3.5.** Let  $A$  be a noetherian commonly graded algebra, and let  $M$  be a finite left  $A$ -module.

- (1) [1] The *cohomological dimension* of  $M$  and of  $A$  are given as, respectively,

$$\begin{aligned} \text{cd}(A) &:= \max\{i \mid R^i\Gamma_{\mathfrak{m}_A}(N) \neq 0 \text{ for some } N \in A\text{-Grm}\} \\ \text{cd}(M) &:= \max\{i \mid R^i\Gamma_{\mathfrak{m}_A}(M) \neq 0\}. \end{aligned}$$

- (2) [10, Proposition 4.3] The *depth* of  $M$  is defined to be

$$\text{dep}(M) := \min\{i \mid R^i\Gamma_{\mathfrak{m}_A}(M) \neq 0\}.$$

- (3)  $M$  is called *Cohen-Macaulay* (CM, or  $d$ -CM) if  $d := \text{cd}(M) < \infty$  and  $\text{dep}(M) = \text{cd}(M)$ .

- (4)  $M$  is called *Maximal Cohen-Macaulay* (MCM) if  $\text{cd}(A) < \infty$  and  $M$  is  $\text{cd}(A)$ -CM.

- (5)  $A$  is called *Cohen-Macaulay* if it is MCM as a left and right  $A$ -module.

Note that when  $A$  is commutative, noetherian, and connected graded (or local), the definition of CM above is equivalent to the standard definition of CM using the Ext-group or using the maximal length of regular sequences [6, Chapter 2].

We recall a few more definitions.

**Definition 3.6.** [1, Definition 3.2] Let  $A$  be a noetherian commonly graded algebra. We say  $A$  satisfies the *left  $\chi$ -condition*, if  $\underline{\text{Ext}}_A^i(S, M)$  is finite dimensional for every left simple graded  $A$ -module  $S$  and every noetherian left graded  $A$ -module  $M$ . The *right  $\chi$ -condition* is defined similarly. If both left and right  $\chi$  hold, then we say  $A$  satisfies  $\chi$ .

It is not difficult to check that this definition is equivalent to the original definition of  $\chi$  given in [1, Definition 3.2].

The following definition is due to Yekutieli [23] and Van den Bergh [20]. Let  $D^b(A\text{-Mod})$  denote the bounded derived category of left  $A$ -modules. Let  $A^{\text{op}}$  denote the opposite ring of  $A$ , and let  $A^e$  denote the enveloping algebra  $A \otimes A^{\text{op}}$ .

**Definition 3.7.** Let  $A$  be a noetherian algebra.

- (1) [23] A complex  $D \in D^b(A^e\text{-Mod})$  is called a *dualizing complex over  $A$*  if it satisfies the following conditions:
  - (i)  $D$  has finite injective dimension over  $A$  and over  $A^{\text{op}}$ ;
  - (ii) for every  $i$ , the  $i$ -th cohomology  $H^i(D)$  is finite over  $A$  and over  $A^{\text{op}}$ , respectively;
  - (iii) the canonical maps  $A \rightarrow \text{RHom}_A(D, D)$  and  $A \rightarrow \text{RHom}_{A^{\text{op}}}(D, D)$  are isomorphisms in  $D(A^e\text{-Mod})$ .
- (2) [23] Assume that  $A$  is commonly graded. Then a graded dualizing complex  $D \in D^b(A^e\text{-Grm})$  is called *balanced* if

$$R\Gamma_{\mathfrak{m}_A}(D)^* \cong R\Gamma_{\mathfrak{m}_{A^{\text{op}}}}(D)^* \cong A$$

as  $A$ -bimodules.

- (3) [20] A dualizing complex  $D$  over  $A$  is called *rigid* if there is an isomorphism

$$D \cong \text{RHom}_{A^e}(A, D \otimes D^{\text{op}})$$

in  $D(A^e\text{-Mod})$ . Here the left  $A^e$ -module structure of  $D \otimes D^{\text{op}}$  comes from the left  $A$ -module structure of  $D$  and the left  $A^{\text{op}}$ -module structure of  $D^{\text{op}}$ .

Now we have the following existence theorems; the original versions of some of these results were given for connected graded algebras [20]. Some of these results were extended to the semilocal case in [21]. Since commonly graded algebras are graded semilocal by Lemma 3.2(3), one can verify the result below following ideas in [20, 21].

**Theorem 3.8.** *Let  $A$  be a noetherian commonly graded algebra.*

- (1) [23, Corollary 4.14] *If  $A$  is AS regular, then  $A$  has a balanced dualizing complex.*
- (2) [20, Theorem 6.3] *In general,  $A$  has a balanced dualizing complex if and only if  $\text{cd}(A)$  and  $\text{cd}(A^{\text{op}})$  are finite, and  $A$  and  $A^{\text{op}}$  satisfy  $\chi$ .*
- (3) [20, Theorem 6.3] *If  $A$  has a balanced dualizing complex  $D$ , then  $D \cong R\Gamma_{\mathfrak{m}}(A)^*$ .*
- (4) [20, Proposition 8.2(2)] *If  $A$  has a balanced dualizing complex  $D$ , then it is a rigid dualizing complex.*

- (5) [20, Proposition 8.2(1)] *The rigid dualizing complex (and the balanced dualizing complex) over  $A$  is unique up to isomorphism.*
- (6) [20, Theorem 5.1] *Let  $D$  be the balanced dualizing complex over  $A$ . Then*

$$R\Gamma_{\mathfrak{m}}(M)^* = \mathrm{RHom}_A(M, D)$$

*for any  $M \in \mathbf{D}(A\text{-Grm})$ .*

- (7) [24, Theorem 4.2(3,4)] *Let  $D$  be the balanced dualizing complex over  $A$ . Then*

$$\mathrm{cd}(A) = \mathrm{cd}(A^{\mathrm{op}}) = -\min\{i \mid H^i(D) \neq 0\} = \max\{i \mid R^i\Gamma_{\mathfrak{m}}(A) \neq 0\}. \quad \square$$

Next we introduce another analogue of the Gorenstein and regularity properties (similar to [18, Definition 3.3]) by employing the dualizing complexes above.

**Definition 3.9.** Let  $A$  be a noetherian commonly graded algebra.

- (1) We say  $A$  is *generalized AS Gorenstein* if
- (i)  $A$  has finite injective dimension  $d$ ,
  - (ii)  $A$  has a balanced dualizing complex, and
  - (iii)  $R^i\Gamma_{\mathfrak{m}}(A)^* \cong \begin{cases} \Omega & i = d \\ 0 & i \neq d \end{cases}$ , where  $\Omega$  is an invertible graded  $A$ -bimodule.
- (2) We say  $A$  is *generalized AS regular* if  $A$  is generalized AS Gorenstein of finite global dimension  $d$ .

**Remark 3.10.** (1) The original definition of AS regularity (Definition 1.1) requires that  $A$  is connected graded. There is a condition missing in the definition given in [18, Definition 3.3], which is “ $R^i\Gamma_{\mathfrak{m}}(A) = 0$  for all  $i \neq d$ ”. In [18, Definition 3.3], one considers  $\mathbb{N}$ -graded (but not necessarily connected) algebras. Here we are considering algebras that are not necessarily  $\mathbb{N}$ -graded.

- (2) If  $A$  is generalized AS Gorenstein, by Theorem 3.8(2-6) the rigid dualizing complex over  $A$  is  $\Omega[d]$ .
- (3) In general,  $\Omega$  is not isomorphic to  $A(-\ell)$  as a graded left  $A$ -modules. For example, let  $A = B \oplus C$ , where  $B$  and  $C$  are AS regular in the sense of Definition 1.1 with the same global dimension but different AS indices. Then  $\Omega = B(-\ell_B) \oplus C(-\ell_C)$ , which cannot be isomorphic to a shift of  $A$  as a graded left  $A$ -module.

We end this section by providing preliminary lemmas about depth, CM modules and MCM modules that are needed in the rest of the paper. We also compare the notion of regularity above with AS regularity as in Definition 1.1. Note that global dimension of the smash product algebra  $A \# H$  is equal to the global dimension of  $A$  when  $H$  is semisimple.

**Lemma 3.11.** *Let  $A$  be a noetherian commonly graded algebra and let  $M$  be a finite left  $A$ -module. Let  $B$  be a subring of  $A$  such that  ${}_B A$  and  $A_B$  are finite  $B$ -modules. Take  ${}_B M$  to be the left  $B$ -module corresponding to  $M$ .*

- (1) [1, Theorem 8.3] We have that  $R^i\Gamma_{\mathfrak{m}_B}({}_B M) = R^i\Gamma_{\mathfrak{m}_A}(M)$ . As a consequence,  $\text{cd}(A) \leq \text{cd}(B)$ .
- (2) The module  ${}_B M$  is CM if and only if  ${}_A M$  is. If  $\text{cd}(A) = \text{cd}(B) < \infty$ , then  ${}_B M$  is MCM if and only if  ${}_A M$  is MCM.
- (3) Suppose  $M = M_1 \oplus M_2$  where  $M_i \neq 0$ . Then  $M$  is CM (respectively, MCM) if and only if both  $M_1$  and  $M_2$  are CM (respectively, MCM) of the same depth.
- (4) If  $A$  is connected graded AS regular, then  $\text{cd}(A) = \text{injdim}(A)$  and  $A$  is MCM.
- (5) If  $A$  is generalized AS regular, then  $\text{cd}(A) = \text{injdim}(A)$  and  $A$  is MCM.

*Proof.* (2) This part follows easily from part (1).

(3) This follows from the fact that the functor  $R^i\Gamma_{\mathfrak{m}_A}(-)$  is additive.

(4) This is well-known.

(5) By Definition 3.9(1),  $R^i\Gamma_{\mathfrak{m}_A}(A) = 0$  for all  $i \neq \text{injdim}(A)$  and  $R^i\Gamma_{\mathfrak{m}_A}(A) \neq 0$  when  $i = \text{injdim}(A)$ . The assertion follows from the fact that  $\text{cd}(A) = \text{injdim}(A)$ , due to Theorem 3.8(7).  $\square$

**Proposition 3.12.** *Let  $A$  be noetherian commonly graded.*

- (1) *If  $A$  is AS regular, then  $A$  is generalized AS regular.*
- (2) [18, Remark 3.6] *If  $A$  is connected graded generalized AS regular, then  $A$  is AS regular.*
- (3) [18, Theorem 4.1(b)] *Let  $H$  be a semisimple Hopf algebra acting on a generalized AS regular algebra  $A$ . Then  $A \# H$  is generalized AS regular.*

*Proof.* (1) Each noetherian connected graded AS regular algebra has a balanced dualizing complex of the form  ${}^\mu A^1(-\ell)[d]$  by [23, Corollary 4.14], and is MCM with  $\text{cd}(A) = \text{cd}(A^{\text{op}}) = \text{injdim}(A)$  by Theorem 3.8(7) and Lemma 3.11(4). Hence  $A$  satisfies Definition 3.9.

(2) [18, Remark 3.6] is still valid.

(3) The proof is the same as [18, Theorem 4.1(b)].  $\square$

**Lemma 3.13.** *Suppose  $A$  is a generalized AS regular algebra and  $M$  is a finite graded left  $A$ -module. Then  $M$  is MCM if and only if  $M$  is projective.*

*Proof.* By definition,  $A$  is MCM. By Lemma 3.11(3), every finite projective  $A$ -module is MCM. For the converse, let  $M$  be a MCM  $A$ -module. It is easy to check that  $A$  is depth-homogeneous in the sense of [22, page 521]. Now by the noncommutative version of Auslander-Buchsbaum formula for (not necessarily connected) graded rings [22, Theorem 3.4],  $M$  is projective.  $\square$

**Lemma 3.14.** [20, Corollary 4.8] *Let  $A$  and  $B$  be noetherian commonly graded algebras with balanced dualizing complexes. Suppose  $M$  is an  $(A, B)$ -bimodule that is a finite module over  $A$  and over  $B$ . Then  $R^i\Gamma_{\mathfrak{m}_A}(M) = R^i\Gamma_{\mathfrak{m}_{B^{\text{op}}}}(M)$ , and, as a consequence,  $\text{dep}({}_A M) = \text{dep}(M_B)$ .*  $\square$

**Lemma 3.15.** *Let  $M$  and  $N$  be nonzero finite left  $A$ -modules related by the exact sequence*

$$0 \rightarrow M \rightarrow P_{s-1} \rightarrow P_{s-2} \rightarrow \cdots \rightarrow P_0 \rightarrow N \rightarrow 0.$$

Then  $\text{dep}(M) \geq \min\{\text{dep}(N)+1, \text{dep}(P_0), \dots, \text{dep}(P_{s-2}), \text{dep}(P_{s-1})\}$ . If, further,  $\text{dep}(P_j) \geq s + \text{dep}(N)$  for each  $j$ , then  $\text{dep}(M) = \text{dep}(N) + s$ .

*Proof.* By induction on  $s$ , it suffices to show the result in the case  $s = 1$ . Letting  $P = P_0$  and applying  $R\Gamma_{\mathfrak{m}}(-)$  to the short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow N \rightarrow 0$ , we obtain

$$\cdots \rightarrow R^{i-1}\Gamma_{\mathfrak{m}}(N) \rightarrow R^i\Gamma_{\mathfrak{m}}(M) \rightarrow R^i\Gamma_{\mathfrak{m}}(P) \rightarrow R^i\Gamma_{\mathfrak{m}}(N) \rightarrow R^{i+1}\Gamma_{\mathfrak{m}}(M) \rightarrow \cdots$$

Since  $R^i\Gamma_{\mathfrak{m}}(P) = 0$  for all  $i < \text{dep}(P)$ , we get  $R^i\Gamma_{\mathfrak{m}}(M) \cong R^{i-1}\Gamma_{\mathfrak{m}}(N)$ . The latter is equal to 0 for all  $i \leq \text{dep}(N)$ . So, for  $i = \text{dep}(N) + 1$ , one has

$$0 \neq R^{i-1}\Gamma_{\mathfrak{m}}(N) \subseteq R^i\Gamma_{\mathfrak{m}}(M),$$

which implies that  $\text{dep}(M) = \text{dep}(N) + 1$ , as required.  $\square$

**Lemma 3.16.** *Let  $A$  and  $B$  be noetherian commonly graded algebras equipped with balanced dualizing complexes. Suppose that  $M$  is a finite right  $B$ -module and  $N$  is an  $(A, B)$ -bimodule that is a finite module over  $A$  and over  $B$ . Then*

$$\text{dep}(\text{Hom}_{B^{\text{op}}}(M, N)) \geq \min\{2, \text{dep}(N)\}.$$

*Proof.* Consider a free resolution of the right  $B$ -module  $M$ :

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where  $P_i$  is finite for  $i = 0, 1$ . By applying  $\text{Hom}_{B^{\text{op}}}(-, N)$  to the short exact sequence above, one has the sequences

$$0 \rightarrow \text{Hom}_{B^{\text{op}}}(M, N) \rightarrow \text{Hom}_{B^{\text{op}}}(P_0, N) \rightarrow \text{coker}_1 \rightarrow 0,$$

and

$$0 \rightarrow \text{coker}_1 \rightarrow \text{Hom}_{B^{\text{op}}}(P_1, N) \rightarrow \text{coker}_2 \rightarrow 0.$$

Since  $P_i$  is projective over  $B$ ,  $\text{Hom}_{B^{\text{op}}}(P_i, N)$  has (left) depth at least equal to  $\text{dep}(N_B)$ , which also equals  $\text{dep}(N_B)$  by Lemma 3.14. Without loss of generality assume that  $\text{dep}(N) \geq 1$  and  $\text{dep}(\text{Hom}_{B^{\text{op}}}(P_0, N)) \geq 1$ . If  $\text{Hom}_{B^{\text{op}}}(P_0, N)$  has depth 1, then, by Lemma 3.15  $\text{dep}(\text{Hom}_{B^{\text{op}}}(M, N)) \geq 1$ , as claimed. Now if  $\text{dep}(N) \geq 2$ , then  $\text{dep}(\text{Hom}_{B^{\text{op}}}(P_i, N)) \geq 2$ . It follows from the short exact sequences above and Lemma 3.15 that  $\text{dep}(\text{coker}_1) \geq 1$  and  $\text{dep}(\text{Hom}_{B^{\text{op}}}(M, N)) \geq 2$ , as required.  $\square$

**Lemma 3.17.** *Let  $A$  be a noetherian AS Gorenstein algebra. Let  $B$  be a graded subalgebra of  $A$  such that  $A_B$  and  ${}_B A$  are finite. Then there is a duality*

$$\text{add}_{\text{grm-}B} A \cong \text{add}_{B\text{-grm}} A.$$

*Proof.* Let  $d$  be the injective dimension of  $A$  and  $\ell$  be the AS index of  $A$ . We will show that the duality functor is  $F := R^d\Gamma_{\mathfrak{m}_B}(-)^*(\ell)$ . By Lemma 3.11(1), if  $M$  is a left graded  $A$ -module, then  $R^d\Gamma_{\mathfrak{m}_A}(M)^* = R^d\Gamma_{\mathfrak{m}_B}(M)^*$ . Hence

$$F({}_B A) = R^d\Gamma_{\mathfrak{m}_A}(A)^*(\ell) \cong A_B,$$

where  $A_B$  is viewed as a graded right  $B$ -module. Thus  $F$  is a functor from  $\text{add}_{B\text{-grm}} A$  to  $\text{add}_{\text{grm-}B} A$ . Similarly,  $G := R^d\Gamma_{\mathfrak{m}_{B^{\text{op}}}}(-)^*(\ell)$  is a functor from



$\text{add}_{\text{grm}-B} A$  to  $\text{add}_{B-\text{grm}} A$ . It is clear that  $GF({}_B A) = {}_B A$  and  $FG(A_B) = A_B$ . Therefore  $F$  induces a duality  $\text{add}_{\text{grm}-B} A \cong \text{add}_{B-\text{grm}} A$ .  $\square$

Theorem A now follows from Proposition 2.3, Corollary 2.6 and Lemma 3.17.

#### 4. A NONCOMMUTATIVE VERSION OF HERZOG'S THEOREM

A theorem of Herzog states that if  $G$  is a finite subgroup of  $\text{GL}(2, \mathbb{k})$  acting linearly on  $\mathbb{k}[x_1, x_2]$ , then the indecomposable MCM  $\mathbb{k}[x_1, x_2]^G$ -modules are precisely the indecomposable  $\mathbb{k}[x_1, x_2]^G$ -direct summands of  $\mathbb{k}[x_1, x_2]$  [9] (or, see [15, Section 6.1]). Here we establish a noncommutative version of this nice result.

Consider the following terminology.

**Definition 4.1.** Let  $A$  be a noetherian ring.

- (1) A subring  $B \subseteq A$  is called a *splitting subring* of  $A$  if  $A_B$  and  ${}_B A$  are finite and if  $B$  is a direct summand of  $A$  as a  $B$ -bimodule.
- (2) When  $B$  is a splitting subring of  $A$ , the projection  $\Phi : A \rightarrow B$  is a  $B$ -bimodule homomorphism called the *Reynolds operator*.

Note that as a consequence of (1),  $B$  is noetherian. Consider the following lemmas about splitting subrings.

**Lemma 4.2.** [12, Lemma 2.4] *If  $H$  is a semisimple Hopf algebra, with  $A$  a noetherian left  $H$ -module algebra, then  $A^H$  is a splitting subring of  $A$ , where the Reynolds operator  $\Phi$  is given by the action of the integral of  $H$  on  $A$ .*  $\square$

**Lemma 4.3.** *Let  $A$  be a noetherian commonly graded algebra. Suppose that  $B$  is a splitting subring of  $A$ .*

- (1) *We have that  $\text{cd}(B) = \text{cd}(A)$ .*
- (2)  *$B$  satisfies  $\chi$  if and only if  $A$  does.*
- (3)  *$B$  has balanced dualizing complex if and only if  $A$  does.*
- (4) *If  $A$  is CM, then so is  $B$ .*
- (5) *If  $A$  is CM, then  $A$  is MCM as a left  $B$ -module.*

*Proof.* (1) By Lemma 3.11(1),  $\text{cd}(A) \leq \text{cd}(B)$ . Since  $R\Gamma_{\mathfrak{m}_A}(A) = R\Gamma_{\mathfrak{m}_B}(A)$ , again by Lemma 3.11(1), we have that  $R\Gamma_{\mathfrak{m}_B}(B)$  is a direct summand of  $R\Gamma_{\mathfrak{m}_A}(A)$ . Therefore

$$\max\{i \mid R^i\Gamma_{\mathfrak{m}_A}(A) \neq 0\} \geq \max\{i \mid R^i\Gamma_{\mathfrak{m}_B}(B) \neq 0\}.$$

By Theorem 3.8(7),  $\text{cd}(A) \geq \text{cd}(B)$ , so the assertion follows.

(2) We adapt the proof of [1, Proposition 8.7]. By [1, Theorem 8.3(2)], if  $B$  satisfies  $\chi$ , so does  $A$ . Conversely, we assume that  $A$  satisfies  $\chi$ . Let  $M$  be a noetherian graded left  $B$ -module. We claim that  $M$  satisfies the  $\chi$ -condition, namely, that  $\underline{\text{Ext}}_B^i(S, M)$  is finite dimensional for all  $i$ , for all simple left  $B$ -modules  $S$ . Let  $N = A \otimes_B M$ . Since  ${}_B M$  is finite, so is  ${}_A N$ . Thus  ${}_A N$  and  ${}_B N$  are noetherian graded modules. Say  $A$  decomposes as  $B \oplus C$  as  $B$ -bimodules, then the Reynolds operator  $\Phi$  induces a decomposition of left  $B$ -modules  ${}_B N = {}_B M \oplus ({}_B C \otimes_B M)$ .

It suffices to show that  ${}_B N$  satisfies the  $\chi$ -condition. By a change of rings spectral sequence,

$$E_2^{p,q} := \underline{\text{Ext}}_A^p(\underline{\text{Tor}}_q^B(A, {}_B S), {}_A N) \Rightarrow \underline{\text{Ext}}_B^{p+q}(S, {}_B N),$$

one sees that  ${}_A N$  satisfies  $\chi$  implies that  ${}_B N$  satisfies  $\chi$ .

(3) This follows from parts (1,2) and Theorem 3.8(2).

(4) The assertion follows from the fact  $R^i \Gamma_{\mathfrak{m}_B}(B)$  is a direct summand of  $R^i \Gamma_{\mathfrak{m}_B}(A) \cong R^i \Gamma_{\mathfrak{m}_A}(A)$ .

(5) The assertion follows from part (1) and Lemma 3.11(2).  $\square$

This brings us to the main result of this section.

**Theorem 4.4.** *Let  $A$  be a generalized AS regular algebra of global dimension 2, and let  $B$  be a splitting subring of  $A$ . Then, up to isomorphism and a degree shift, the indecomposable MCM left  $B$ -modules are precisely the indecomposable  $B$ -direct summands of  ${}_B A$ .*

An example of this result was discussed in [11, Section 2.3].

*Proof of Theorem 4.4.* Since  $A$  is generalized AS regular, it is CM and MCM by Lemma 3.11(5). Hence  $A$  is left MCM over  $B$  by Lemma 4.3(5), and every  $B$ -direct summand of  ${}_B A$  is MCM over  $B$ , as well, by Lemma 3.11(3).

To begin the other direction, let  $D_A$  be the balanced dualizing complex over  $A$ , which exists by Theorem 3.8(2). Let  $d = \text{cd}(A)$ . Then  $D_A \cong P[d]$ , where  $P = R^d \Gamma_{\mathfrak{m}_A}(A)^*$  by Theorem 3.8(3). Let  $D_B$  be the balanced dualizing complex over  $B$ , which exists by Lemma 4.3(3). It follows that  $D_B$  has the form  $\Omega[d]$ , where  $\Omega = R^d \Gamma_{\mathfrak{m}_B}(B)^*$  by Theorem 3.8(3).

For every left MCM  $B$ -module  $M$ , define  $M^\vee$  to be

$$\begin{aligned} M^\vee &:= \text{RHom}_B(M[d], D_B) = \text{RHom}_B(M, D_B)[-d] \\ &= R^d \Gamma_{\mathfrak{m}_B}(M)^* = \text{Hom}_B(M, \Omega), \end{aligned}$$

where the last two equalities hold using local duality by Theorem 3.8(6). Then

$$(M^\vee)^\vee \cong \text{Hom}_{B^{\text{op}}}(M^\vee[d], D_B) \cong \text{Hom}_{B^{\text{op}}}(\text{RHom}_B(M, D_B), D_B) \cong M.$$

Denote by  $C$  the kernel of the Reynolds operator  $\Phi : A \rightarrow B$ . Applying  $R^d \Gamma_{\mathfrak{m}_B}(-)^*$  to  $\Phi$  gives a split exact sequence of  $B$ -bimodules

$$(E4.4.1) \quad 0 \rightarrow \Omega \rightarrow P \rightarrow C^\vee \rightarrow 0.$$

Applying  $\text{Hom}_{B^{\text{op}}}(M^\vee, -)$  to (E4.4.1) gives a split exact sequence of left  $B$ -modules

$$0 \rightarrow \text{Hom}_{B^{\text{op}}}(M^\vee, \Omega) \rightarrow \text{Hom}_{B^{\text{op}}}(M^\vee, P) \rightarrow \text{Hom}_{B^{\text{op}}}(M^\vee, C^\vee) \rightarrow 0.$$

This shows that  $M \cong \text{Hom}_{B^{\text{op}}}(M^\vee, \Omega)$  is a  $B$ -direct summand of the left  $A$ -module  $\text{Hom}_{B^{\text{op}}}(M^\vee, P)$ .

By the theory of dualizing complexes [23, Proposition 3.4],  $M^\vee$  is a finite right  $B$ -module. Since  $P$  is invertible, the left  $A$ -module  ${}_A P$  is a progenerator; so,  $\text{dep}(P) = \text{dep}(A) = 2$ . By Lemma 3.16,  $Y := \text{Hom}_{B^{\text{op}}}(M^\vee, P)$  is a noetherian left  $A$ -module of depth at least 2. But the maximal depth of  $Y$  is 2, as  $A$  has global dimension 2. Therefore, by definition,  $Y$  is MCM over  $A$ . Since  $A$  is generalized AS regular

of global dimension 2,  $Y$  is projective as a left  $A$ -module by Lemma 3.13. Since  $M \cong \text{Hom}_{B^{\text{op}}}(M^\vee, \Omega)$  is a  $B$ -direct summand of  $Y$ ,  $M$  is a direct summand of a projective left  $A$ -module. If  $M$  is indecomposable, then, up to a degree shift, it is a direct summand of  ${}_B A$ .  $\square$

**Corollary 4.5.** *Let  $H$  be a semisimple Hopf algebra. Suppose that  $A$  is an  $H$ -module algebra satisfying the hypotheses of Theorem 4.4. Then, up to isomorphism and a degree shift, the indecomposable MCM left  $A^H$ -modules are precisely the indecomposable direct summands of  $A$  as a left  $A^H$ -module.*

*Proof.* By Lemma 4.2, take  $A^H$  to be the splitting subring of  $A$ . Now apply Theorem 4.4.  $\square$

Theorem C follows immediately from Corollary 4.5.

## 5. A NONCOMMUTATIVE VERSION OF LEUSCHKE'S THEOREMS

In this section we prove noncommutative versions, Theorem D (Theorem 5.4) and Theorem E (Theorem 5.7), of results of Leuschke [13, Theorem 6 and Proposition 8]. These results pertain to the global dimension and the *representation dimension* (Definition 5.5) of endomorphism rings of MCM modules.

We begin with the following preliminary result. Recall that in the notation  $\text{End}_B(M)$  or  $\underline{\text{End}}_B(M)$ , the module  $M$  is considered as a right  $B$ -module.

**Lemma 5.1.** *Let  $B$  be an algebra that is module-finite over a central affine graded subalgebra  $C$ , and let  $M$  be a finite right  $B$ -module. Then  $\text{End}_B(M)$  is a noetherian PI algebra. If further,  $B$  is commonly graded and  $M$  is a graded right  $B$ -module, then  $\underline{\text{End}}_B(M)$  has a balanced dualizing complex.*

*Proof.* Note that  $C$  is a noetherian commutative algebra. Since  $C$  is central in  $B$  there is a natural map from  $C$  to  $\text{End}_B(M)$ . Since  $B$  is finite over  $C$ , so is  $M$ . Applying  $\text{Hom}_{B^{\text{op}}}(-, M)$  to a surjective map  $B^{\oplus n} \rightarrow M$ , one sees that  $\text{End}_B(M)$  is a submodule of  $\text{Hom}_{B^{\text{op}}}(B^{\oplus n}, M) \cong M^{\oplus n}$ , which is finite over  $C$ . Hence  $\text{End}_B(M)$  is finite over  $C$ . Therefore  $\text{End}_B(M)$  is a noetherian PI ring.

Since  $B$  and  $M$  are graded, so is  $\underline{\text{End}}_B(M)$ . A graded version of [21, Corollary 0.2] implies that  $\underline{\text{End}}_{B^{\text{op}}}(M)$  has a balanced dualizing complex.  $\square$

Now the main results of this section hold for algebras of *finite CM type*, which we define next.

**Definition 5.2.** We say that a (left) noetherian commonly graded algebra  $A$  is of *finite Cohen-Macaulay (finite CM) type* if  $A$  has, up to a degree shift, finitely many non-isomorphic indecomposable MCM modules.

**Example 5.3.** Let  $A$  be a connected graded or generalized AS regular algebra of dimension 2 that admits an action of a semisimple Hopf algebra  $H$ . Then the fixed subring  $B := A^H$  is of finite CM type by Corollary 4.5. Note that the  $B$ -module  $A$  has only finitely many direct summands.

Next we prove Theorem D.

**Theorem 5.4.** *Let  $B$  be a noetherian commonly graded algebra such that  $B$  has a balanced dualizing complex. Suppose that  $B$  is CM and of finite CM type. Let  $M$  be a finite direct sum of MCM right  $B$ -modules that contain at least one copy of each MCM module, up to a degree shift. Let  $E := \text{End}_B(M)$ .*

- (1) *If  $E$  is right noetherian, then  $E$  has right global dimension at most  $\max\{2, d\}$ , where  $d = \text{cd}(B)$ .*
- (2) *If  $d \geq 2$ , then the right global dimension of  $E$  is equal to  $d$ .*

*Proof.* (1) We modify the proof given in [13, Theorem 6]. The case  $d = 1$  is easy, so we assume that  $d \geq 2$ .

Let  $X = X_E$  be a finite right  $E$ -module, and consider the first  $d - 1$  steps in the projective resolution of  $X_E$ :

$$(E5.4.1) \quad P_\bullet : P_{d-1} \xrightarrow{f_{d-1}} P_{d-2} \rightarrow \cdots \rightarrow P_1 \xrightarrow{f_1} P_0$$

with  $X = \text{coker } f_1$ . Since  $E$  is right noetherian by assumption, we may assume that each  $P_i$  is a finite projective right  $E$ -module. By Proposition 2.5(3),  $P_i = \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_i)$  for some  $N_i \in \text{add}_{\text{grm-}B^{\text{op}}} M$ . By Lemma 3.11(3) each  $N_i$  is a MCM right  $B$ -module. By Proposition 2.5(3),  $f_i = \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, g_i)$ , where  $g_i \in \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(N_i, N_{i-1})$ . Putting these facts together, we have the following sequence of morphisms between MCM graded right  $B$ -modules:

$$N_\bullet : N_{d-1} \xrightarrow{g_{d-1}} N_{d-2} \rightarrow \cdots \rightarrow N_1 \xrightarrow{g_1} N_0$$

such that  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_\bullet) = P_\bullet$ .

Since  $B$  is CM, every indecomposable direct summand of  $B$  is an MCM module, which must be a direct summand of  $M$ . Hence  $M = M_0 \oplus M_1$ , where  $M_0$  is a graded progenerator of  $B$ . Therefore  $P_\bullet = \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_\bullet)$  contains a  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M_0, N_\bullet)$  as a direct summand. Since  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_\bullet)$  is exact, so is  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M_0, N_\bullet)$ . Note that  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M_0, -)$  is an equivalence of categories by Morita theory. Therefore,  $N_\bullet$  is exact. Let  $N_d$  be the kernel of  $g_{d-1}$ . Then we have an exact sequence

$$(E5.4.2) \quad 0 \rightarrow N_d \rightarrow N_{d-1} \xrightarrow{g_{d-1}} N_{d-2} \rightarrow \cdots \rightarrow N_1 \xrightarrow{g_1} N_0.$$

By Lemma 3.15,  $N_d$  is a MCM. Because  $M$  contains all indecomposable MCM modules,  $N_d$  is in  $\text{add}_{\text{grm-}B^{\text{op}}} M$ . Applying  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, -)$  to (E5.4.2), one has an exact sequence

$$0 \rightarrow \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_d) \rightarrow P_{d-1} \xrightarrow{f_{d-1}} P_{d-2} \rightarrow \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \rightarrow X \rightarrow 0.$$

Since  $N_d \in \text{add}_{\text{grm-}B^{\text{op}}} M$ ,  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_d) \in \text{grprn-}E$  by Proposition 2.5(2). Therefore  $X$  has projective dimension at most  $d$ . This shows that  $E$  has right global dimension at most  $d$ .

Next we show that the right global dimension of  $E$  is  $d$  if  $d \geq 2$ . Let  $Y$  be a nonzero finite dimensional right  $B$ -module (so,  $\text{dep}(Y) = 0$ ), and consider an exact sequence

$$N_1 \xrightarrow{g_1} N_0 \xrightarrow{g_0} Y \rightarrow 0,$$

where  $N_0$  and  $N_1$  are projective  $B$ -modules. Let  $X$  be the cokernel of the map  $\underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, g_1)$ , so we have a partial projective resolution  $P_1 \rightarrow P_0 \rightarrow X$ , where  $P_i = \underline{\text{Hom}}_{\text{grm-}B^{\text{op}}}(M, N_i)$ . Extend this to a projective resolution (E5.4.1). We claim that the projective dimension of  $X$  is  $d$ . If not, we may assume that  $P_d = 0$ . By the construction, as before, we have an exact sequence  $N_\bullet$  with  $N_d = 0$ . By Lemma 3.15,  $\text{dep}(Y) \geq \text{dep}(N_{d-1}) - (d-1) > 0$ , which contradicts the fact that  $\text{dep}(Y) = 0$ .

(2) This follows exactly as in the proof of [13, Theorem 6].  $\square$

**Definition 5.5.** Let  $A$  be a noetherian CM commonly graded algebra of depth  $d$  and let  $\omega = R^d\Gamma_{\mathfrak{m}}(A)^*$ . The *representation dimension* of  $A$  is defined to be

$$\text{repdim}(A) = \inf\{\text{gldim}(\text{End}_A(A \oplus \omega \oplus M)) : M \text{ is a MCM left } A\text{-module}\}.$$

We obtain the following generalization of a result of Leuschke [13, Proposition 8] as one immediate consequence of Theorem 5.4.

**Corollary 5.6.** *Let  $B$  be a noetherian, CM, commonly graded algebra, of finite CM type, that is module-finite over a central affine graded subalgebra. Then we have that  $\text{repdim}(B) \leq \max\{2, \text{cd}(B)\}$ .*

*Proof.* Let  $E := \text{End}_B(B \oplus R^{\text{cd}(B)}\Gamma_{\mathfrak{m}}(B)^* \oplus N)$ , where  $N$  is the direct sum of the remaining indecomposable MCM left  $B$ -modules. Then  $E$  is noetherian by Lemma 5.1, and hence,  $E$  has global dimension at most  $\max\{2, \text{cd}(B)\}$  by Theorem 5.4.  $\square$

There are stronger results for graded CM algebras  $B$ , of finite CM type, when  $\text{cd}(B) = 2$ . We begin with a proof of Theorem E, which is an easy consequence of Theorem 5.4.

**Theorem 5.7.** *Let  $B$  be a noetherian commonly graded algebra that is CM with  $\text{cd}(B) = 2$ , of finite CM type, and module-finite over a central affine graded subalgebra  $C$ . Let  $M$  be a finite direct sum of MCM right  $B$ -modules that contains at least one copy of each MCM module, up to a degree shift. Then  $E := \underline{\text{End}}_B(M)$  is a noetherian PI generalized AS regular algebra of dimension 2.*

*Proof.* By Theorem 5.4,  $E$  has right global dimension 2. By Lemma 5.1,  $E$  is noetherian PI, equipped with a balanced dualizing complex. Hence,  $E$  has left global dimension 2. By Lemma 3.15, the depth (and thus cohomological dimension) of  $E$  is 2, or equivalently,  $R^i\Gamma_{\mathfrak{m}}(E) = 0$  for  $i = 0, 1$ .

Let  $\Omega$  be  $R^2\Gamma_{\mathfrak{m}}(E)^*$ . It remains to show that  $\Omega$  is invertible. By Theorem 3.8(3),  $D := \Omega[2]$  is the balanced dualizing complex over  $E$ . Then

$$E = \text{RHom}_E(D, D) \cong R\Gamma_{\mathfrak{m}}(D) \cong R\Gamma_{\mathfrak{m}}(\Omega)^*[-2].$$

Therefore  $R^i\Gamma_{\mathfrak{m}}(\Omega)^* = 0$  for  $i = 0, 1$ , and  $R^2\Gamma_{\mathfrak{m}}(\Omega)^* = E$ . Thus  $\Omega$  is MCM. Since  $E$  is MCM, it is depth-homogeneous in the sense of [22, page 521]. By [22, Theorem 3.4],  $\Omega$  is a projective (left and right)  $E$ -module. By the discussion before the proof of [25, Theorem 0.2, page 561], since  $E$  is Gorenstein,  $D$  is also a two-sided tilting complex over  $E$ , or equivalently, an invertible complex over  $E$  by [25, Definition 2.1]. Since  $D$  is a stalk complex  $\Omega[2]$  that is invertible, we obtain that  $\Omega$  is an invertible  $E$ -bimodule. Therefore  $E$  is generalized AS regular.  $\square$

**Corollary 5.8.** *Let  $B$  be noetherian commonly graded algebra that is CM with  $\text{cd}(B) = 2$ , of finite CM type, and module-finite over a central affine graded subalgebra  $C$ . Then the center  $Z$  of  $B$  is reduced.*

*Proof.* Let  $M$  be a finite direct sum of all MCM right  $B$ -modules that contain at least one copy of each MCM module, up to a degree shift, and a copy of  $B$ . Then the central action of  $Z$  on  $M$  is faithful. Let  $E := \text{End}_{B^{\text{op}}}(M)$ . By Theorem 5.7,  $E$  is generalized AS regular and is PI. By [19, Theorem 1.4(1)],  $E$  is semiprime. Hence, any central subring of  $E$ , including  $Z$ , is reduced.  $\square$

**Remark 5.9.** Theorem 5.7 and Corollary 5.8 apply to the fixed subring  $B = A^H$  arising from an action of a semisimple Hopf algebra  $H$  on an AS regular algebra  $A$  of dimension 2; see Example 5.3.

Theorem 5.7 fails for  $\text{cd}(B) \geq 3$ , even in the commutative case, as the following example demonstrates.

**Example 5.10.** [13, Example 11] Let  $B$  be the subring of  $A = \mathbb{k}[x, y, z]$  generated by degree two elements, namely,  $B = \mathbb{k}[x^2, xy, y^2, xz, yz, z^2]$ . (The subring  $B$  arises as the fixed subring of  $A$  under the  $\mathbb{Z}_2$ -action given by negation.) Then  $B$  has cohomological dimension 3 (e.g. by Lemma 3.15), and  $B$  has three MCM graded modules, up to degree shift and isomorphism. When  $M$  is the direct sum of these three MCM modules, we obtain that  $E := \underline{\text{End}}_B(M)$  has global dimension 3. However,  $E$  also has depth 2 (which can be checked with Macaulay 2, as mentioned in [13]). Therefore  $E$  is not CM, and hence not AS regular by Lemma 3.11(4).

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